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Constructing QCD Loop Amplitudes Using Collinear Limits*

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Abstract

We discuss how higher-point QCD amplitudes may be constructed from lower point ones by imposing the factorization constraints in the limits as external momenta become collinear. As a particular example, the all-n gluon one-loop amplitude with maximal helicity violation is presented. We also discuss the necessary collinear behavior of the n-gluon amplitudes.

1. Introduction. The task of finding new physics in future experiments requires an accurate understanding of the perturbative QCD background. In jet physics, for example, one needs the loop corrections in order to reduce theoretical uncertainties in the cone angle and the renormalization scale dependence. However, the algebraic complexity of loop calculations in gauge theories prohibits the practical use of conventional techniques for large numbers of partons. Even a four-point one-loop amplitude is formidable with conventional Feynman diagrammatic techniques [1]. Indirect means may alternatively be used to find amplitudes, without any Feynman diagrams. For example, at tree-level concise formulae for maximally helicity violating amplitudes with an arbitrary number of external legs were first conjectured by Parke and Taylor [2] and later proven by Berends and Giele [3] using recursion relations. The result was found in part by examining the collinear behavior of tree-level amplitudes, the universal behavior of which is independent of the number of external legs.

At one-loop order the QCD gluon amplitudes have a natural decomposition in terms of supersymmetric contributions plus a complex scalar loop piece [4,5]. The supersymmetric parts are most easily obtained [6] from unitarity through the use of Cutkosky rules. The scalar contribution, however, contains polynomials which cannot be determined in the same manner. Our goal here is to show how the universal behavior of the loop level collinear limits may be used to find scalar pieces of one-loop gluon amplitudes.

The consistency conditions which one-loop gauge theory amplitudes satisfy in the limits when the external momenta become collinear or soft are strong enough to determine certain amplitudes without using any Feynman diagrams. In this talk we discuss an example of a one-loop amplitude which is sufficiently constrained that we can write down a form for an arbitrary number of external

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legs. The particular all-n result which we present [7] is for maximal helicity violation, that is with all (outgoing) legs of identical helicity, and has since been confirmed by recursive techniques [8]. The construction is based upon extending the known one-loop four- and five-gluon [4] amplitudes which were obtained using string-based methods [9].

In order to use the collinear limits to fix the polynomial terms, one needs a proof that the scalar contributions to the loop have a universal behavior for any number of external legs as the momenta of two of the legs become collinear. We discuss such a proof here. Since the cuts can be used to determine the amplitudes completely except for the polynomial part of the scalar loop piece, we only focus here on the collinear limits of these scalar loop contributions.

2. Review. We first briefly review some of the important tools relevant to the content of the present work: color-ordering the non-abelian Feynman rules, the use of a spinor helicity basis, and the development of string-improved methods.

Color ordering the Feynman rules amounts to separating a given amplitude into independent color structures, which must then be individually gauge invariant. In the adjoint representation and at tree level, we may rewrite the structure constants in all the vertices as $f_{abc} = -i/\sqrt{2}\mathrm{Tr}\Big([T^a, T^b], T^c\Big)$ and by using trace identities write the full n-gluon amplitude as [10]:

$$\mathcal{A}_n^{\mathrm{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\ A_n^{\mathrm{tree}}(k_{\sigma(1)}^{\lambda_{\sigma(1)}}, \ldots, k_{\sigma(n)}^{\lambda_{\sigma(n)}})\ , \tag{1}$$

where k_i, λ_i, and a_i are respectively the momentum, helicity ($\pm$), and color index of the i-th external gluon. The coupling constant is g, and S_n/Z_n is the set of non-cyclic permutations of $\{1, \ldots, n\}$. When calculating the full amplitude we only have to concentrate on finding the simpler color-ordered partial amplitudes A_n. The full contribution is simply a sum over all inequivalent orderings of the color traces times ordered amplitudes.

The corresponding color ordering for the adjoint representation at loop level is slightly more complicated [11], giving:

$$\mathcal{A}_n\left(\{k_i, \lambda_i, a_i\}\right) = g^n \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}\left(\sigma\right)\ A_{n;c}^{[J]}(\sigma), \tag{2}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and n_J is the number of particles of spin J. The leading color-structure factor, $\mathrm{Gr}_{n;1}(1) = N_c\ \mathrm{Tr}\left(T^{a_1} \cdots T^{a_n}\right)$, is just the number of colors, N_c, times the tree color factor. The subleading color structures ($c > 1$) are given by $\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}\left(T^{a_1} \cdots T^{a_{c-1}}\right) \mathrm{Tr}\left(T^{a_c} \cdots T^{a_n}\right)$, S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant. For internal particles in the fundamental ($N_c + \bar{N}_c$) representation, only the single-trace color structure ($c = 1$) would be present, and the corresponding

color factor would be smaller by a factor of N_c. In this talk we concern ourselves only with the partial amplitudes $A_{n;1}$; the other partial amplitudes $A_{n;c}$ may be obtained by appropriate permutations over $A_{n;1}$ [6].

Another advance in the technology of amplitude calculations is in the use of a spinor helicity basis [12]. Roughly speaking, there is a large degree of redundancy in the terms of Feynman diagrams contributing to an amplitude. Much of this redundancy can be removed by utilizing the invariance of the amplitude under $\varepsilon_i \to \varepsilon_i + f(k_i)k_i$, and choosing appropriate gauges. In the spinor helicity basis all quantities are written in terms of Weyl spinors $|k^{\pm}\rangle$, which provide a convenient shorthand for expressing results. In this formalism, the polarization vectors are expressed as $\varepsilon^+_\alpha(p,q) = \langle q^-|\gamma_\alpha|p^-\rangle/\sqrt{2}\langle q^-|p^+\rangle$ and $\varepsilon^-_\alpha(p,q) = \langle q^+|\gamma_\alpha|p^+\rangle/\sqrt{2}\langle p^+|q^-\rangle$. The important point is that the reference momenta q, like $f(k_i)$ above, reflects the gauge invariance of the amplitude and must drop out of any final result (so choose it advantageously in a given calculation). For the purposes of this presentation we note that

$$\langle k_i^-|k_j^+\rangle \equiv \langle ij\rangle = \sqrt{2k_i \cdot k_j}\,\exp(i\phi), \qquad \langle k_i^+|k_j^-\rangle \equiv [ij] = \sqrt{2k_i \cdot k_j}\,\exp(-i\phi) \tag{3}$$

which vanishes in the limit $k_i \cdot k_j \to 0$.

More recently, methods based on limits of certain string theories have advanced the ability to do one-loop perturbative QCD calculations [9]. This technique has been used in the calculation of all five-point gluon helicity amplitudes, which are necessary as a starting point to generate the all-n amplitude given in this talk.

3. Collinear Limits. We intend to use the collinear factorization properties of amplitudes to constrain the scalar part of the n-gluon helicity amplitudes. First we discuss these particular kinematical limits of the amplitudes. The collinear limits of color-ordered one-loop QCD amplitudes (with spin-J particle content in the loop) are expected to have the form:

$$A_{n;1}^{[J]} \xrightarrow{a\|b} \sum_{\lambda=\pm}\left(\mathrm{Split}^{\mathrm{tree}}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\,A^{[J]}_{n-1;1}(\ldots(a+b)^\lambda\ldots) + \mathrm{Split}^{[J]}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\,A^{\mathrm{tree}}_{n-1}(\ldots(a+b)^\lambda\ldots)\right) \tag{4}$$

in the limit where the momenta $k_a \to zP$ and $k_b \to (1-z)P$ with $P = k_a + k_b$ (where $b = a+1$). Here λ is the helicity of the intermediate state with momentum P, and the splitting functions $\mathrm{Split}_{-\lambda}$ describe the divergent infrared behavior. This is analogous to the form of tree-level collinear limits [13,2]. All known one-loop amplitudes satisfy eq. (4). Due to the supersymmetry decomposition and techniques based on unitarity, our main interest is in proving the factorization for the case of scalars in the loop. For the example of the all-plus helicity amplitude, the scalar collinear limits are sufficient because a SUSY Ward identity relates the gluon and fermion contribution to the scalar one (discussed further below).

In the following, all of the Feynman diagrams contributing to the scalar contribution $A_{n;1}^{[0]}$ are systematically studied with the aim of showing that the functions $\mathrm{Split}^{[0]}$ are independent of n. The diagrams are first categorized into several sets depending upon the topology of the two external collinear legs. The conclusion is that: (a) $\mathrm{Split}^{\mathrm{tree}}$ arises from the diagrams in fig. 1, (b) $\mathrm{Split}^{[0]}$ from the diagrams in fig. 2, and (c) diagrams without *explicit* $1/k_1 \cdot k_2$ poles such as in fig. 3 contribute nothing to the scalar loop splitting functions. The forthcoming analysis is broken up accordingly.

a. Contributions to $\mathrm{Split}^{\mathrm{tree}}$*.* Diagrams where the adjacent collinear legs are located in an external tree to the scalar loop potentially contribute to $\mathrm{Split}^{\mathrm{tree}}$. The exceptional case of this classification, fig. 2b, contributes to $\mathrm{Split}^{[0]}$. It is not difficult to see that the only tree Feynman diagrams which contain poles in $1/(k_1+k_2)^2$, and hence are leading order in the collinear limit, are those which have a two-particle external tree as in fig. 1. The hatched circle represents the loop amplitude $A_{n-1;1}^{[0]}\big(1+2,\ldots\big)$. Out of this set we acquire the tree splitting functions, which are given in [13], but repeated here for convenience:

$$A^{\text{fig. 1a}} \longrightarrow^{k_1\|k_2} \sum_{\lambda=\pm 1} \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(1^{\lambda_1},2^{\lambda_2}) \quad A_{n-1;1}^{[0]}\big(\ldots,(1+2)^{\lambda},\ldots\big)$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}(1^+,2^+) = \frac{1}{\sqrt{z(1-z)}\langle 12\rangle}, \qquad \mathrm{Split}^{\mathrm{tree}}_{-}(1^+,2^-) = -\frac{z^2}{\sqrt{z(1-z)}[12]},$$
$$\mathrm{Split}^{\mathrm{tree}}_{-}(1^-,2^+) = -\frac{(1-z)^2}{\sqrt{z(1-z)}[12]}, \qquad \mathrm{Split}^{\mathrm{tree}}_{-}(1^-,2^-) = \mathrm{Split}^{\mathrm{tree}}_{+}(1^+,2^+) = 0\,. \tag{5}$$

b. Contributions to $\mathrm{Split}^{[0]}$*.* Diagrams with the collinear legs attached to a loop, but with an explicit collinear pole give rise to the $\mathrm{Split}^{[0]}$ contribution. The sum of these diagrams, given in figs. 2a-c, give a result which does not have any singularity as k_1 or $k_2 \to 0$, does not require renormalization, and only contains a collinear divergence. Together they contain the entire contribution to the loop splitting functions for internal scalars. We note that the finiteness (in $1/(d-4)$) follows by the Ward identity which relates the infinities of the two- and three-point functions. We have from figs. 2a-c,

$$\begin{aligned} A^{\text{fig 1}}(1,2,\ldots) &\longrightarrow^{k_1\|k_2} \frac{1}{16\pi^2}\frac{1}{6}\Big(\varepsilon_1\cdot\varepsilon_2 - \frac{\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1}{k_1\cdot k_2}\Big)\Big(\frac{1}{\sqrt{2}k_1\cdot k_2}\Big)(k_1-k_2)^{\mu}A^{tree}_{n-1\mu}(1+2,\ldots,n) \\ &\equiv \sum_{\lambda=\pm} \mathrm{Split}^{[0]}_{-\lambda}(1^{\lambda_1},2^{\lambda_2})\ A^{\mathrm{tree}}_{n-1}\big((1+2)^{\lambda},\ldots,n\big)\,, \end{aligned} \tag{6}$$

and after using a helicity basis similar to that in ref. [3]:

$$\mathrm{Split}^{[0]}_{-}(1^+,2^+) = \frac{1}{48\pi^2}\sqrt{z(1-z)}\frac{1}{\langle 12\rangle}, \qquad \mathrm{Split}^{[0]}_{+}(1^+,2^+) = \frac{-1}{48\pi^2}\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle^2},$$

$$\mathrm{Split}^{[0]}_{\pm}(1^+,2^-) = \mathrm{Split}^{[0]}_{\pm}(1^-,2^+) = 0\,. \tag{7}$$

c. Subleading as $k_1 \cdot k_2 \to 0$. We now argue that the remaining diagrams do not have any leading collinear poles. We first classify the diagrams according to:

(1) One or both of the collinear legs are attached via a four-vertex with one of its neighbors (not a collinear partner) to the scalar loop. Alternatively, one collinear leg may be directly connected to a loop where the other collinear leg is part of a tree sewn onto the loop.

(2) Both collinear legs are attached to a scalar loop by three-point vertices and are part of a loop with four or more legs as depicted in fig. 3a.

(3) The collinear legs are attached by a four-point vertex to a loop as in fig. 3b.

The diagrams in set (1) do not have poles for two reasons. First, the scalar product $k_1 \cdot k_2$ is prevented to appear in the loop integration just by the structure of the momentum flow. Second, the external trees attached to these loops do not have any internal lines that vanish in the collinear limit. There are no s_{12} channels in any tree attached to a loop of this type.

All the loop diagrams in sets (2) and (3) are also found to give subleading contributions in the collinear limit. Consider the one-loop diagram $D_n = \mu^\varepsilon (g/\sqrt{2})^n G_n$ with all external gluon legs connected directly to the scalar loop by three-point vertices:

$$G_n = \int \frac{d^d l}{(2\pi)^d} \left(\prod_{j=1}^{n} \frac{1}{l_j^2} \right) \left(\prod_{k=1}^{n} 2\varepsilon_k \cdot l_k \right) \qquad \text{with} \;\; l_j = l - q_j, \;\; q_j \equiv \sum_{a=2}^{j} k_a \;(a \bmod n)\,. \tag{8}$$

The leading $1/k_1 \cdot k_2$ collinear singularities of the integrand come from a surface of loop momentum ($l = ak_1 + bk_2$) with thickness of the order $k_1 \cdot k_2$. In this region three propagators simultaneously blow up. (The two points $l = k_2$ and $l = -k_1$ must be considered separately since a fourth propagators blows up in the integrand, but the conclusions are the same.)

We examine the integral by first rewriting the integration in a manner to extract the contribution from the surface spanning k_1 and k_2. This is accomplished by breaking the loop momentum into three components $l = l_\perp + \alpha k_1 + \beta k_2$ ($l_\perp \cdot k_1 = l_\perp \cdot k_2 = 0$), with the measure changing as $d^4 l = d^2 l_\perp (2k_1 \cdot k_2) d\alpha d\beta$ – valid in Minkowski space. Since the scalar one-loop diagrams are infrared finite no dimensional regulator will be used. In this manner, with $l_\perp \cdot k_1 = l_\perp \cdot k_2 = 0$

$$G_n = \frac{1}{(2\pi)^4} \int d^2 l_\perp d\alpha d\beta (2k_1 \cdot k_2) \left(\prod_{j=1}^{n} \frac{1}{(l_\perp + \alpha k_1 + \beta k_2 - q_j)^2} \right) \left(\prod_{k=1}^{n} 2\varepsilon_k \cdot (l_\perp + \alpha k_1 + \beta k_2 - q_j) \right). \tag{9}$$

The potential collinear divergence in the denominator comes from the three propagators adjacent to and between the first and second legs, and arises when $l_\perp^2$ becomes small with respect to the squared momentum flowing through the three propagators.

In order to extract a collinear singularity from the denominator of the integral, the component perpendicular to the surface, $l_\perp^2$, has to be restricted to roughly $k_1 \cdot k_2$ – the 'width' of the surface spanned by $\ell = \alpha k_1 + \beta k_2$. In euclidean space we define a cutoff so that $l_\perp^2 \leq \Lambda^2 \sim k_1 \cdot k_2$. To leading order, we approximate the expression for G_n by ignoring $l_\perp$ in all but the propagators $j = n, 1, 2$ in eq. (9), and in all vertices $l_\perp \cdot \varepsilon_i$ (with $i \neq 1, 2$); the corrections are found in a Taylor expansion in $l_\perp$, but these terms are suppressed since $l_\perp^2/q_j^2 \sim k_1 \cdot k_2/q_j^2$. In this way we obtain a 'triangle' of three propagators which contributes the potential collinear behavior.

The contribution to the loop diagram (9) from the plane spanned by k_1 and k_2 in this approximation is found by integrating over $l_\perp$ within a region $l_\perp^2 \leq \Lambda^2$,

$$G_n \approx -\frac{i}{(2\pi)^4} \int d\alpha d\beta \quad (2k_1 \cdot k_2) \left(\prod_{j \neq n,1,2}^{n} \frac{1}{(\alpha k_1 + \beta k_2 - q_j)^2} \right) \left(\prod_{k \neq 1,2}^{n} 2\varepsilon_k \cdot (\alpha k_1 + \beta k_2 - q_j) \right)$$
$$\times \frac{\pi}{k_1 \cdot k_2} \Big(\varepsilon_1 \cdot \varepsilon_2 g(\alpha, \beta, \Lambda'^2) - \frac{\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1}{k_1 \cdot k_2} f(\alpha, \beta, \Lambda'^2) \Big) . \tag{10}$$

The functions f and g are dimensionless quantities and $\Lambda'^2 \equiv \Lambda^2/2k_1 \cdot k_2 \sim O(1)$. By a dimensional argument or through direct integration, we see that there are no leading singularities in either f or g.

Since there are no leading collinear divergences along the plane of loop momenta spanned by k_1 and k_2, we conclude that these graphs do not contribute to the loop splitting functions in the limit $k_1 \cdot k_2 \to 0$. The same analysis may be applied to any other graph in this class and to those in the third set.

A modified analysis for the case in which fermions are in the loop give the same conclusions as for the scalars. For the gluons in the loop, however, it turns out that the diagrams in fig. 3a do contain collinear singularities; also there are additional complications from infrared singularities. A more detailed discussion will be presented elsewhere. As previously mentioned, for purposes of constructing higher-point amplitudes via collinear limits, only the scalar loop contribution is required. Hence only the scalar splitting functions are necessary.

4. All-Plus Amplitude with Arbitrary Numbers of Legs. The all-plus helicity amplitude $A_{n;1}$ is particularly simple; it is cyclicly symmetric, and no logarithms or other functions containing branch cuts can appear. This can be seen by considering the cutting rules: the cut in a given channel is given by a phase space integral of the product of the two tree amplitudes obtained from cutting. One of these tree amplitudes will vanish for all assignments of helicities on the cut internal legs since $A_n^{\text{tree}}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0$, so that in fact all cuts vanish. Similar reasoning shows that the all plus helicity loop amplitude does not contain multi-particle poles; factorizing the amplitude on

a multi-particle pole into lower point tree and loop amplitudes again yields a tree which vanishes for either helicity of the leg carrying the multi-particle pole.

Another simplifying feature of the all-plus amplitude is the equality, up to a sign due to statistics, of the contributions of internal gluons, complex scalars and Weyl fermions. This is a consequence of the supersymmetry Ward identity [14] $A^{\rm susy}(1^\pm, 2^+, \ldots, n^+) = 0$ for $N = 1$ and $N = 2$ theories. Since the $N = 1$ supersymmetry amplitude has one gluon and one gluino circulating in the loop, the gluino contribution must be equal and opposite to that of the gluon in order to yield zero for the total; similarly, the spectrum of an $N = 2$ supersymmetric theory contains two gluinos and one complex scalar in addition to the gluon, and the vanishing implies the equality of the contributions of complex scalars and gluons circulating in the loop (i.e. $A^{[0]} = A^{[1]} = -A^{[1/2]}$).

The starting point in constructing our n-point expression is the known five-point one-loop helicity amplitude [4],

$$A_{5;1}(1^+, 2^+, 3^+, 4^+, 5^+) \;=\; \frac{iN_p}{192\pi^2} \frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} \;+\; \varepsilon(1, 2, 3, 4)}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,1\rangle}\,, \tag{11}$$

where $\varepsilon(1, 2, 3, 4) = 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma \;=\; [1\,2]\,\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle - \langle 1\,2\rangle\,[2\,3]\,\langle 3\,4\rangle\,[4\,1]$, and N_p is the number of color-weighted bosonic states minus fermionic states circulating in the loop; for QCD with n_f quarks, $N_p = 2(1 - n_f/N)$ with $N = 3$.

Using eq. (4) and $A_n^{\rm tree}(1^\pm, 2^+, \cdots, n^+) = 0$, we can construct higher point amplitudes by writing down general forms with only two particle-poles, and requiring that they have the correct collinear limits. To start the procedure one assumes that the denominator for the six-point amplitude is $\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle 6\,1\rangle$. The numerator is a polynomial of the correct dimensions with coefficients fixed by the collinear limit constraints to the five-point amplitude (11). Continuing in this way we can generalize to the all-n result [7]

$$A_{n;1}(1^+, 2^+, \ldots, n^+) \;=\; -\frac{iN_p}{192\pi^2} \sum_{1\le i_1<i_2<i_3<i_4\le n} \frac{{\rm tr}[(1+\gamma_5)\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}]}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle}\,, \tag{12}$$

a form which has been confirmed in [8]. It is straightforward to verify that the above all-plus amplitude satisfies the collinear limits given in eq. (4).

Additionally, the collinear limits of loop amplitudes leads one to suspect that many amplitudes in massless QED and mixed QCD/QED must vanish, as found for the all-plus amplitude in QED by Mahlon [8]. First, charge conjugation invariance implies that photon amplitudes with an odd number of legs vanish. This also implies that the amplitude with three photons and two gluons $A_{5;1}(\gamma_1, \gamma_2, \gamma_3, g_4, g_5) = 0$, since this amplitude is proportional to the corresponding photon amplitude: the two gluons have to be in a color singlet. Using the collinear behavior (4) leads one to suspect that the six-point all-plus helicity amplitudes (lacking cuts) with three photons and three

gluons may vanish, and continuing recursively in this way, that perhaps all amplitudes with three photons and additional gluons vanish. However, it is possible to construct functions containing logs which have non-singular behavior in all collinear limits [6]; these non-singular functions are 'missed' in the collinear bootstrap and pose a potential problem to acquiring log and dilog terms through the collinear approach. For our case, the amplitudes are free of cuts so that these type of functions cannot appear, although one might still worry about cut free functions with no singularities in any channel. Nevertheless, we can directly verify that the all-plus amplitudes with three or more photons vanish.

Amplitudes with r external photons and $(n-r)$ gluons have a color decomposition similar to that of the pure-gluon amplitudes, except that charge matrices are set to unity for the photon legs. The coefficients of these color factors, $A^{r\gamma}_{n;1}$, are given by appropriate cyclic sums over the pure-gluon partial amplitudes. One can write down simple forms for the all-plus partial amplitude with one or two external photons, and any number of gluons [7]. By explicitly performing the sum over permutations, we find that for three or more external photons the amplitude vanishes,

$$A^{\rm loop}_{n>4}(\gamma_1^+,\gamma_2^+,\gamma_3^+,g_4^+,\ldots,g_n^+) = 0. \tag{13}$$

Since amplitudes with even more photon legs are obtained by further sums over permutations of legs, the all-plus helicity amplitudes with three or more photon legs vanish (for $n>4$) in agreement with the expectation from the collinear limits. The same result holds when one of the helicities is reversed [15].

5. Conclusions. The combined usage of collinear limits and unitarity provides powerful methods for further calculations of gauge theory amplitudes. On the one hand, the Cutkosky rules fix the logarithmic parts to amplitudes and miss the contributions of rational functions (those which have no cuts). The QCD n-gluon amplitudes can be decomposed into supersymmetric plus scalar loop contributions; indeed, it has been proved that the cuts uniquely determine the supersymmetric parts of the gluon one-loop amplitudes [6]. The scalar contributions contain polynomial terms which are not uniquely fixed by the cuts, and the collinear limit approach is effective in constraining these contributions. In this way, these two approaches complement each other.

Collinear limits also provide a strong check on amplitudes obtained by other means, such as string-improved [4], recursive [8], or unitarity techniques [6]. We expect that the approach based on the collinear limits will be a useful tool for finding further one-loop amplitudes (e.g. $n=6,7,\ldots$) for general helicity configurations.

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Figure Captions.

Fig. 1: Diagrams that contribute to the tree splitting functions.

Fig. 2: Diagrams that contribute to the loop splitting functions.

Fig. 3: Two of the remaining diagram types which have no collinear poles for scalars in the loop.

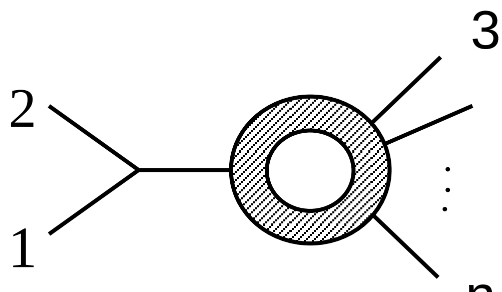

Fig. 1

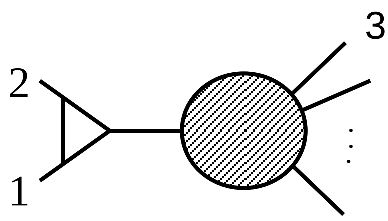

Fig. 2a

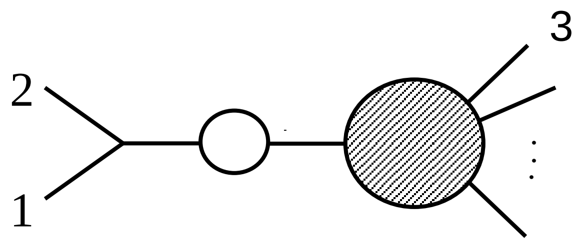

Fig. 2b

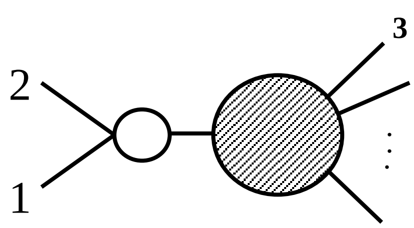

Fig. 2c

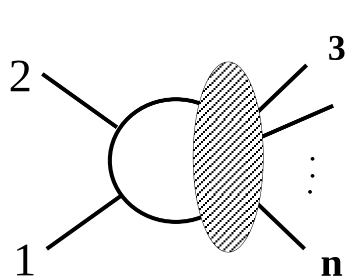

Fig. 3a

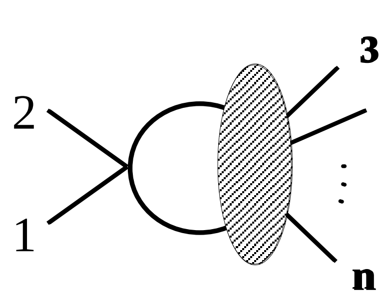

Fig. 3b